Revisit: FTC Part 1 – Special Case

constant in upper bound

$$\frac{d}{dx}\int_{x^3}^{5} t^2\,dt = -\frac{d}{dx}\int_{5}^{x^3} t^2\,dt = (x^3)^2 \cdot (x^3)' = x^6 \cdot 3x^2 = \boxed{3x^8}$$

x in lower bound

can reverse bounds of an integral by negating the integral $\int_a^b f(x)dx = -\int_b^a f(x)dx$

recall: $\int x^2 dx = \frac{x^3}{3} + C$

$\int \frac{1}{x} dx = \ln|x| + C$

Recall: Differentials

$$\boxed{\frac{dy}{dx} = f'(x)}$$

$dy = f'(x)\,dx$

ex. Find the differential of: $y = x^3$

$$\boxed{(ax^n)' = nax^{n-1}}$$

$\frac{dy}{dx} = 3x^2 \Rightarrow dy = 3x^2\,dx$

Substitution Rule

Use the Substitution Rule to integrate when there are ~~2~~ Two functions within the integral AND one is the derivative of the other.

ex. Find $\int 2x\sqrt{1+x^2}\,dx$

$$\boxed{\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad \text{where } n \neq -1}$$

$= \int \sqrt{1+x^2}\,\underbrace{2x\,dx}$ ← $\boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}}$

$(1+x^2)' = 2x$ ←

$\frac{d}{dx}(1+x^2) = 2x$

$= \int \sqrt{u}\;\;du$ ← now we have a basic integral

$= \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C$ ← convert back to be ITO x

$= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$

Examples when *u* must first be manipulated:

ex. Find $\int \underbrace{(x^3+1)}_{u}^5 \cdot \underbrace{3x^2\,dx}$

$u = x^3+1$

$du = 3x^2\,dx$

$= \int u^5 \, du$

$= \frac{u^6}{6} + C$

$= \boxed{\frac{(x^3+1)^6}{6} + C}$

ex. find $\int (x^3+1)^5 \underbrace{x^2\,dx}$

$u = x^3+1$

$\frac{1}{3} du = \frac{\cancel{3}}{\cancel{3}} x^2\,dx$

$\frac{1}{3} du = x^2\,dx$

$= \int u^5 \cdot \frac{1}{3} du$

$= \frac{1}{3} \int u^5 \, du$

$= \frac{1}{3} \cdot \frac{u^6}{6} + C$

$= \boxed{\frac{1}{18}(x^3+1)^6 + C}$

OR $\frac{(x^3+1)^6}{18} + C$

ex. Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ $= \int \frac{1}{\sqrt{1-4x^2}} \cdot x\,dx$

$u = 1-4x^2$

$du = -8x\,dx$

$-\frac{1}{8} du = x\,dx$

$\frac{a}{b} = \frac{1}{b} \cdot \frac{a}{1}$

$= -\frac{1}{8} \int \frac{1}{\sqrt{u}} \, du$

$= -\frac{1}{8} \int u^{-1/2} \, du$

$\int u^{-1/2} du = \frac{u^{1/2}}{1/2}$

$= -\frac{1}{\cancel{8}_4} \cdot \cancel{2} u^{1/2} + C$

$u^{1/2} = \sqrt{u}$

$= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}$

composed function

ex. Find $\int e^{5x}\,dx$

$u = 5x$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$= \frac{1}{5}\int e^u\,du$

$= \frac{1}{5}e^u + C$

$= \frac{1}{5}e^{5x} + C$

check by differentiating: $\left(\frac{1}{5}e^{5x} + C\right)'$

$= \frac{1}{\cancel{5}}\cdot\cancel{5}e^{5x} \quad 0$

$= e^{5x}$ ✓

Do: Find $\int e^{x/2}\,dx = \int e^{x/2}\,dx$

$u = \frac{1}{2}x$

$du = \frac{1}{2}dx$

$2du = dx$

$= 2\int e^u\,du$

$= 2e^u + C \Rightarrow \boxed{2e^{x/2} + C}$

$\int \frac{1}{u^1}du = \int u^{-1}du = \frac{u^0}{\cancel{0}}$ UND ←

Do: Find $\int \frac{x}{1-6x^2}\,dx$

$u = 1-6x^2$

$du = -12x\,dx$

$-\frac{1}{12}du = x\,dx$

$= \int \frac{1}{1-6x^2}\cdot x\,dx$

$= -\frac{1}{12}\int \frac{1}{u}\,du$

$= -\frac{1}{12}\ln|u| + C$

$= \boxed{-\frac{1}{12}\ln|1-6x^2| + C}$

$(\ln x)' = \frac{1}{x}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

Integrate trigonometric functions using the Substitution Rule:

ex. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$= \int \frac{1}{\cos x} \cdot \underline{\sin x \, dx}$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\int \frac{1}{u} \, du$

$= -\ln|u| + C$

$= \boxed{-\ln|\cos x| + C}$

ex. $\int \cot x \csc x \, dx$

Hint: rewrite trig functions ITO sine, cosine

$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \, dx$

$= \int \frac{1}{(\sin x)^2} \cdot \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int \frac{1}{u^2} \, du$

$= \int u^{-2} \, du$

$= -u^{-1} + C = -\frac{1}{u} + C = \boxed{-\frac{1}{\sin x} + C}$

A different situation with u:

ex. Find $\int \sqrt{2x+1} \, dx$

$u = 2x+1$

$du = 2\,dx$

$\frac{1}{2} du = dx$

$= \int \sqrt{u} \left(\frac{1}{2} du\right)$

$= \frac{1}{2} \int u^{1/2} \, du$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$

$= \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$

Substitution Rule for Definite Integrals

if g' is continuous on $[a,b]$ and if f is continuous where $u=g(x)$

then $\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)}$

$= F(g(b)) - F(g(a))$

i.e. bounds must reflect the variable used for integration

Revisit: Evaluate $\int_{0=x}^{4=x} \sqrt{2x+1}\,dx$

$u = 2x+1$, $\frac{1}{2}du = dx$ from prev. pg.

convert x-bounds to u-bounds

$u_{x=0} = 2(0)+1 = 1$

$u_{x=4} = 2(4)+1 = 9$

$= \frac{1}{2}\int_{u=1}^{u=9} \sqrt{u}\,du$

$= \frac{1}{3}u^{3/2}\Big|_{u=1}^{u=9}$

$9^{3/2} = (9^{1/2})^3 = \sqrt{9}^3 = 3^3 = 27$

$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{1}{3}(27-1) = \boxed{\frac{26}{3}}$

ex. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(3-5x)^2}\,dx$

$u = 3-5x$

$du = -5dx \Rightarrow -\frac{1}{5}du = dx$

$u_{x=1} = 3-5 = -2$

$u_{x=2} = 3-5(2) = -7$

$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du$

$= -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du$

$= -\frac{1}{5}(-1)u^{-1}\Big|_{-2}^{-7}$

$= \frac{1}{5}\cdot\frac{1}{u}\Big|_{-2}^{-7}$ $\quad F(b)-F(a)$

$= \frac{1}{5}\left(-\frac{1}{7} - \left(-\frac{1}{2}\right)\right)$

$= \frac{1}{5}\left(-\frac{1}{7}\cdot\frac{2}{2} + \frac{1}{2}\cdot\frac{7}{7}\right)$

$= \frac{1}{5}\left(\frac{7}{14} - \frac{2}{14}\right)$

$= \frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{14} = \boxed{\frac{1}{14}}$

$(\ln x)' = \frac{1}{x}$

ex. Evaluate $\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \frac{1}{x}\cdot \ln x\,dx$

$u = \ln x \rightarrow u_{x=1} = \ln 1 = 0$

$u = \ln x \rightarrow u_{x=e} = \ln e = 1$

$du = \frac{1}{x}\,dx$

$$= \int_1^e \ln x \cdot \frac{1}{x}\,dx$$

$$= \int_0^1 u\,du$$

$$= \frac{u^2}{2}\Big|_0^1$$

$$= \frac{1}{2}\cdot u^2\Big|_0^1$$

$$= \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$$